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AN INTEGRAL PROCEDURE FOR A TWO-PHASE BOUNDARY LAYER IN LAMINAR FILM CONDENSATION

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Abstract—A general integral procedure has been developed for the analysis of laminar film condensation heat transfer. A solution of the gravity force condensation problem with constant properties is presented, including the effect of the drag due to an initially stationary body of pure saturated vapor. The present integral method, based on the two-phase boundary layer theory, assumes a finite density–viscosity ratio between the liquid and vapor phases, and can, in fact, be employed under almost all possible two dimensional and axisymmetric geometrical configurations. An extensive comparison of the calculation results with available exact solutions reveals an excellent performance of the present integral procedure.

INTRODUCTION

Koh *et al.* (1961) attacked the body force film condensation problem by simultaneously solving the complete liquid and vapor boundary layer equations through the similarity transformation. Subsequently, Koh (1961) proposed an integral method and solved the same problem for the limiting condition, namely, an infinite density–viscosity ratio $R = \rho\mu/(\rho\mu)_G$ (where ρ is the liquid density; μ is its viscosity; thus, the quantities associated with the liquid phase will be presented without a subscript, while those associated with the vapor phase will be identified by the subscript G).

The present paper proposes an integral procedure for obtaining the solution to the two-phase boundary layer equations in the laminar film condensation in the presence of a body force. The analysis differs from previous approximate analyses by retaining not only the inertia and convection terms, but also the density–viscosity ratio R as an additional parameter so as to investigate their effects on the heat transfer functions. Thus, it is valid even for the case of low Prandtl number fluids at a high pressure, in which R is not significantly greater than unity (note, R approaches unity at the critical pressure). Moreover, the method can be employed for almost all possible plane and axisymmetric bodies. A comparison of the flat plate calculation results with the exact solutions, in fact, reveals an excellent performance of the present integral solution procedure.

PHYSICAL MODEL AND GOVERNING EQUATIONS

The physical model and boundary layer coordinates (x, y) are indicated in figure 1. The body may be plane or axisymmetric, and its wall geometry is given by the function $r(x)$. The wall surface at a constant temperature T_w is exposed to a quiescent, pure vapor at the saturation temperature T_i ($>T_w$). Thus, condensation occurs on the isothermal wall. Consequently, there simultaneously develop both liquid and vapor boundary layers.

A usual control volume consideration within the liquid film thickness δ and the vapor layer thickness Δ leads to the following momentum equations:

Liquid:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \rho \overset{*}{r} u^2\,\mathrm{d}y - u_i\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \rho \overset{*}{r} u\,\mathrm{d}y = \overset{*}{r}\left[(\rho-\rho_G)g_x\delta - \mu\left.\frac{\partial u}{\partial y}\right|_w + \mu\left.\frac{\partial u}{\partial y}\right|_i\right]. \quad [1]$$

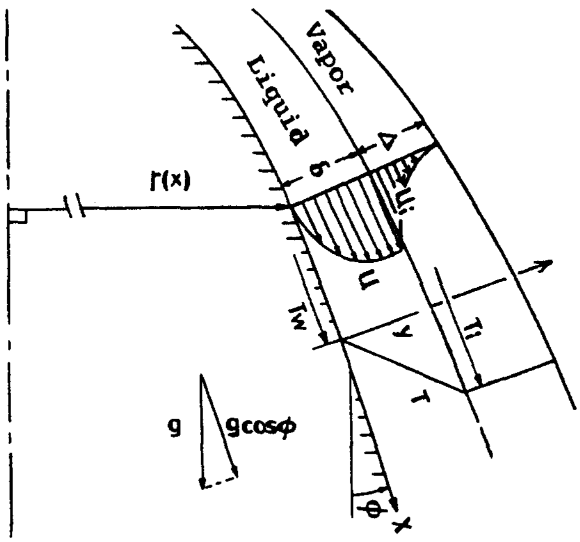

Figure 1. Physical model and coordinate system.

Vapor:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\delta}^{\delta+\Delta} \rho_G \overset{*}{r} u^2 \,\mathrm{d}y + u_i \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\delta} \rho \overset{*}{r} u \,\mathrm{d}y = -\overset{*}{r}\mu \left.\frac{\partial u}{\partial y}\right|_i, \quad [2]$$

where

$$\overset{*}{r} = \begin{cases} 1: & \text{plane flow,} \\ r(x): & \text{axisymmetric flow,} \end{cases} \quad [3a]$$

and

$$g_x \equiv g\cos\phi = g\left[1 - \left(\frac{\mathrm{d}r}{\mathrm{d}x}\right)^2\right]^{1/2}, \quad [3b]$$

where g is the acceleration due to gravity, u is the steamwise velocity component, ϕ is the surface orientation angle, while the subscripts w and i refer to the wall ($y = 0$) and the liquid–vapor interface ($y = \delta$), respectively. In addition to the no-slip condition for the velocity, the following compatibility conditions on the liquid–vapor interface have been already implemented in the vapor momentum equation [2]:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\delta} \rho \overset{*}{r} u \,\mathrm{d}y = \rho_G \overset{*}{r}\left(u_i \frac{\mathrm{d}\delta}{\mathrm{d}x} - v_i\right)_G \quad [4]$$

and

$$\mu \left.\frac{\partial u}{\partial y}\right|_i = \mu_G \left.\frac{\partial u}{\partial y}\right|_{i,G}, \quad [5]$$

where v_i is the normal direction velocity component at the interface.

The integral equation for the energy conservation within the liquid phase may be given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \rho \overset{*}{r} u Cp(T - T_i)\,\mathrm{d}y = \overset{*}{r} k\left(\left.\frac{\partial T}{\partial y}\right|_i - \left.\frac{\partial T}{\partial y}\right|_w\right). \qquad [6]$$

Here, T is the local temperature, Cp is the specific heat of the liquid, and k is its thermal conductivity. Furthermore, the following energy balance relation should hold along the interface:

$$h_{LG}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \rho \overset{*}{r} u\,\mathrm{d}y = \overset{*}{r} k\left.\frac{\partial T}{\partial y}\right|_i \qquad [7]$$

where h_{LG} is the latent heat of condensation. Since the advection terms in the momentum and energy equations must vanish near the wall, the conservation equations in their original differential form indicate

$$\Lambda \equiv -\frac{1}{2}\frac{\delta^2}{u_i}\left.\frac{\partial^2 u}{\partial y^2}\right|_w = \frac{g_x'\delta^2}{2\nu u_i} \qquad [8a]$$

and

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_w = 0, \qquad [8b]$$

where

$$g_x' \equiv (\rho - \rho_G)g_x/\rho. \qquad [8c]$$

Upon considering the auxiliary equations above, the velocity and temperature profiles within the liquid layer may be prescribed as

$$f(\eta;\Lambda,C) \equiv u/u_i = C\eta - \Lambda\eta^2 + (1 - C + \Lambda)\eta^3 \qquad [9a]$$

and

$$\theta(\eta;\Lambda_t) \equiv (T - T_i)/(T_w - T_i) = 1 - (1 + \Lambda_t)\eta + \Lambda_t\eta^3, \qquad [9b]$$

where

$$\eta \equiv y/\delta. \qquad [9c]$$

In addition to [8a, b], the velocity profile satisfies $f = 0$, $\partial f/\partial\eta = C$ at $\eta = 0$ and $f = 1$ at $\eta = 1$, while the temperature profile meets the conditions, $\theta = 1$, $\partial\theta/\partial\eta = -(1 + \Lambda_t)$ at $\eta = 0$ and $\theta = 0$ at $\eta = 1$. Obviously, the shape factor Λ_t accounts for the nonlinearity of the temperature profile. The velocity profile in the vapor layer, on the other hand, may be specified simply as

$$f_G(\eta_G) \equiv u/u_i = 1 - 2\eta_G + \eta_G^2, \qquad [10a]$$

where

$$\eta_G \equiv (y - \delta)/\Delta. \qquad [10b]$$

Substituting [9a, b] and [10a] into [1], [2], [6], [7] and [5], one obtains

$$\frac{\mathrm{d}}{\mathrm{d}x}B\overset{*}{r}u_i^2\delta - u_i\frac{\mathrm{d}}{\mathrm{d}x}A\overset{*}{r}u_i\delta = \overset{*}{r}\left[g_x'\delta + \nu(3 + \Lambda - 3C)\frac{u_i}{\delta}\right], \qquad [11a]$$

$$\frac{1}{5}(\rho_G/\rho)\frac{\mathrm{d}}{\mathrm{d}x}\overset{*}{r}u_i^2\Delta + u_i\frac{\mathrm{d}}{\mathrm{d}x}A\overset{*}{r}u_i\delta = \frac{\nu E\overset{*}{r}u_i}{\delta}, \qquad [11b]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}D\overset{*}{r}u_i\delta = \frac{3\nu\Lambda_t\overset{*}{r}}{\Pr\delta}, \qquad [11c]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}A\overset{*}{r}u_i\delta = \frac{\nu(1 - 2\Lambda_t)H\overset{*}{r}}{\Pr\delta} \qquad [11d]$$

and

$$\delta = \mu E\Delta/2\mu_G, \qquad [11e]$$

where Pr is the liquid Prandtl number. Moreover, the sensible-latent heat ratio H and the shape factors are defined as

$$H \equiv Cp(T_i - T_w)/h_{LG}, \qquad [12a]$$

$$A \equiv \int_0^1 f\,\mathrm{d}\eta = (3 + 3C - \Lambda)/12, \qquad [12b]$$

$$B \equiv \int_0^1 f^2\,\mathrm{d}\eta = (30 + 24C + 16C^2 - 10\Lambda - 11C\Lambda + 2\Lambda^2)/210, \qquad [12c]$$

$$D = \int_0^1 f\,\theta\,\mathrm{d}\eta = (21 + 49C - 24\Lambda_t - 32C\Lambda_t - 14\Lambda + 11\Lambda\Lambda_t)/420 \qquad [12d]$$

and

$$E \equiv -\left.\frac{\partial f}{\partial\eta}\right|_{\eta=1} = 2C - \Lambda - 3. \qquad [12e]$$

GENERAL SOLUTIONS

The differential equations [11a, b, c, d] coupled to the auxiliary equations [8a] and [11e] should simultaneously be solved for the six unknowns, namely, δ, Δ, u_i, Λ, C and Λ_t. Among these unknowns, the interfacial velocity u_i and the vapor layer thickness Δ may be eliminated in favor of Λ and δ using [8a] and [11e]. For the case of general nonsimilar flows, the shape factors such as Λ and C may vary downstream. However, the rates of the streamwise variations of these shape factors may usually be small enough for the "quasilocal similarity" to hold as already substantiated for the film condensation on the horizontal circular cylinder without interfacial shear (Nakayama & Koyama 1984). Thus, neglecting the first derivatives of these shape factors, one can solve equations [11a, b, c] to obtain three distinct expressions for δ^4 as

$$\left(\frac{\delta}{x}\right)^4 Grx = \frac{24\Lambda(1 + \Lambda - C)}{5B - 3A}I = \frac{8\Lambda RE^2}{3RAE + 2}I_G = \frac{8\Lambda\Lambda_t}{\Pr D}I_t, \qquad [13]$$

where

$$I = \frac{\int_0^x (\overset{*}{r}^{4B-4A} g_x^{3B-A})^{1/(5B-3A)}\,\mathrm{d}x}{(\overset{*}{r}^{4B-4A} g_x^{3B-A})^{1/(5B-3A)}\,x}, \qquad [14a]$$

$$I_G = \frac{\int_0^x (\overset{*}{r}^{4RAE+8/5}\, g_x^{RAE+6/5})^{1/(3RAE+2)}\,\mathrm{d}x}{(\overset{*}{r}^{4RAE+8/5}\, g_x^{RAE+6/5})^{1/(3RAE+2)}\,x}, \qquad [14b]$$

$$I_t = \frac{\int_0^x (\overset{*}{r}^4 g_x)^{1/3}\,\mathrm{d}x}{(\overset{*}{r}^4 g_x)^{1/3}\,x} \qquad [14c]$$

and

$$Grx = g_x' x^3/\nu^2. \qquad [14d]$$

The first and second expressions in the right-hand side of [13] may be combined to give

$$\frac{I_G}{I} = \frac{9A(1 - C + \Lambda)}{E(5B - 3A)}\left(1 + \frac{2}{3RAE}\right). \qquad [15]$$

Upon substituting [12d] into the last expression in [14a], the second and the last right-hand expressions are equated and solved for Λ_t as

$$\Lambda_t = \frac{21 + 49C - 14\Lambda}{24 + 32C - 11\Lambda + 1260\,A(1 + 2/3\,RAE)\,(I_t/I_G E\,\mathrm{Pr})}. \qquad [16]$$

Furthermore, [11c, d] under the assumption of the quasilocal similarity readily lead to the expression for the sensible-latent heat ratio H as

$$H = 3\,A\,\Lambda_t/\mathrm{D}(1 - 2\Lambda_t). \qquad [17]$$

Thus, the solution of the problem has eventually been reduced to the determination of the three unknown shape factors, namely, Λ, C and Λ_t from [15], [16] and [17] for given Pr, H, R and $\overset{*}{r}$. Upon locally evaluating the functions $I(x)$, $I_G(x)$ and $I_t(x)$ according to the prescribed geometry $\overset{*}{r}(x)$, one must simultaneously solve the above three equations for $\Lambda(x)$, $C(x)$ and $\Lambda_t(x)$. The solution, in general, requires an iterative procedure at each integration step. Such a step-wise iterative procedure has already been described in detail for the case without interfacial shear (Nakayama & Koyama 1984). The initial values of the shape factors at $x = 0$, however, must be determined before marching downstream in consideration of the similar film layers as described in the following section. Once the shape factors are determined, the interfacial velocity u_i and Nusselt number $Nux \equiv h\,x/k = (x/(T_i - T_w))(\partial T/\partial y)\,|_w$ (where h is the local heat transfer coefficient) are given by

$$(u_i x/\nu)/Grx^{1/2} = (C_\delta I_t H/\mathrm{Pr})^{1/2}/2\Lambda, \qquad [18a]$$

$$Nux/Grx^{1/4} = (1 + \Lambda_t)(\mathrm{Pr}/H C_\delta I_t)^{1/4}, \qquad [18b]$$

where

$$C_\delta \equiv 8\Lambda(1 - 2\Lambda_t)/3A. \qquad [18c]$$

Thus,

$$(\delta/x)^4 Grx = C_\delta I_t H/\mathrm{Pr}. \qquad [18d]$$

SOLUTIONS TO SIMILAR FILM LAYERS

The consideration of the stagnation flows around $x = 0$ (Koyama & Nakayama 1982) leads to the following facts:

$$\overset{*}{r} \propto x^i, \quad i = \begin{cases} 0: & \text{plane flows,} \\ 1: & \text{axisymmetric flows,} \end{cases} \qquad [19a]$$

$$g_x \propto x^j, \quad j = \begin{cases} 0: & \text{pointed body,} \\ 1: & \text{blunt body.} \end{cases} \qquad [19b]$$

The functions I, I_G and I_t under the relations above reduce to

$$I = \left(1 + \frac{4\,(B - A)\,i + (3B - A)\,j}{5B - 3A}\right)^{-1}, \qquad [20a]$$

$$I_G = \left(1 + \frac{4\,(RAE + 2/5)\,i + (RAE + 6/5)\,j}{3RAE + 2}\right)^{-1} \qquad [20b]$$

and

$$I_t = [1 + (4i + j)/3]^{-1}. \qquad [20c]$$

For example, the integers (i, j) may be given as (0, 0) for a vertical flat plate, (1, 0) for a cone, (0, 1) for the stagnation flow over a horizontal circular cylinder and (1, 1) for the stagnation flow over a sphere. It is interesting to note that, as the density–viscosity ratio R becomes sufficiently large, the function I_G asymptotically approaches to I_t, and the terms associated with R in [15] and [16] vanish at the same time. Consequently, the solutions become fairly insensitive to R as increasing R. Hence, the fact which was pointed out by Koh *et al.* (1961) for a vertical flat plate pertains to all similar condensate layers.

For the similar film layers, the substitution of [20] reduces [15], [16] and [17] to a set of simple algebraic equations. Once Pr, H and R are specified, these three algebraic equations can be solved for Λ, C and Λ_t. In actual calculations, it is more expedient to use an inverse method. First, for a prescribed shape factor C is determined from [15]. Then, Λ_t is readily calculable from [16]. Finally, [17] gives the corresponding sensible-latent heat ratio H. In this way, one can construct the solution curves for the similar film layers.

RESULTS AND DISCUSSION

Figure 2 is plotted to show the effects of H on the shape factors. According to the definitions, the shape factors Λ, C and A are associated with the velocity curvature, gradient and the condensation rate, respectively, while the abscissa variable H is related with the film thickness as may be observed in [18d]. An increase in Pr only brings the shape factors closer to their limiting values corresponding to the Nusselt's solution (Nusselt 1916) namely, $\Lambda = 1$, $C = 2$ and $A = 2/3$. Hence, the effects of the inertia and interfacial shear on the velocity profiles are expected to become negligible for high Prandtl numbers.

In figure 3, the comparison is made among the present solution, the exact solution by Koh *et al.* (1961) and the approximate solutions by Koh (1961) in terms of the local heat transfer rates. The present solution appears to be in excellent agreement with the exact

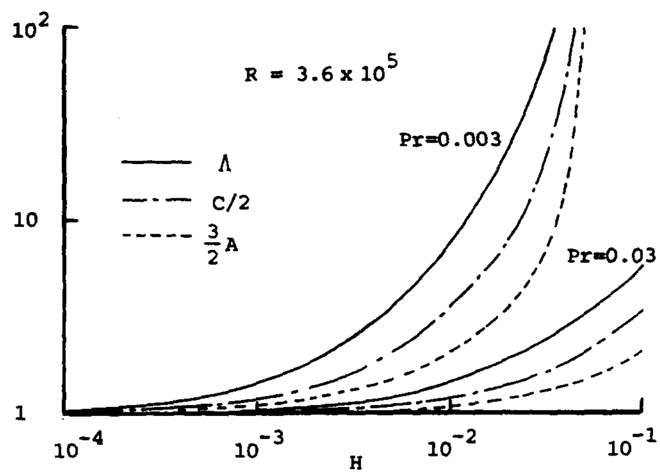

Figure 2. Velocity shape factors.

solution, while Koh's approximate solution based on the assumption of an infinite B shows an appreciable deviation from it.

The predicted velocity profiles are plotted along with the exact solutions in figures 4(a) and 4(b) for high and low Prandtl numbers, respectively. The abscissa variables in the figures are chosen as the similarity variables used by Koh *et al.* (1961), namely,

$$\eta_L = \frac{y}{x}\left(\frac{Grx}{4}\right)^{1/4}$$

and

$$\eta_G = \frac{y-\delta}{x}\left(\frac{\nu}{\nu_G}\right)^{1/2}\left(\frac{Grx}{4}\right)^{1/4}.$$

The figures demonstrate that even the details of the velocity profiles agree closely with the exact solutions.

Figure 5 presents the temperature profiles obtained for $H = 0.702$ and 0.0819, with $\mathrm{Pr} = 10$ and $R = 100$. Again, the predicted profiles turn out to be in good accord with the exact solutions. A careful observation reveals the nonlinearity of the temperature profile due to convection ($\Lambda_t = 0.0717$ for $H = 0.702$ and $\Lambda_t = 0.00997$ for $H = 0.0819$).

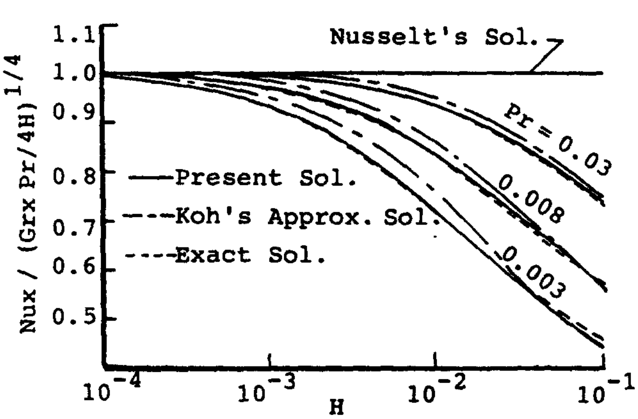

Figure 3. Comparison of approximate and exact solutions.

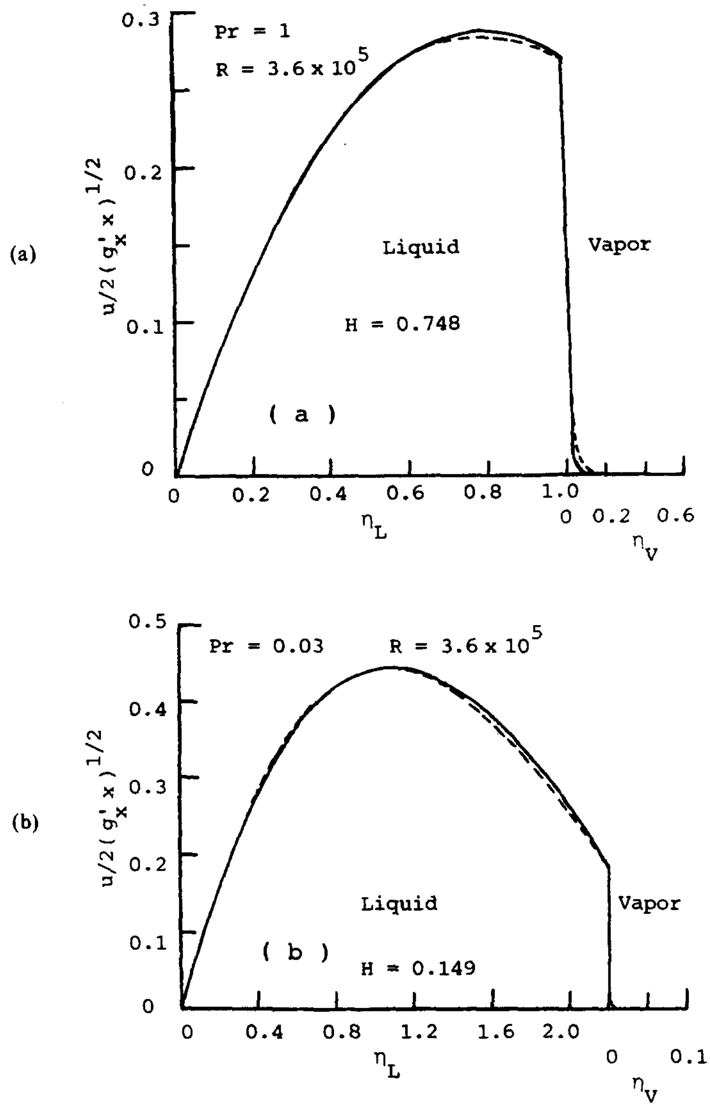

Figure 4. Velocity profiles.

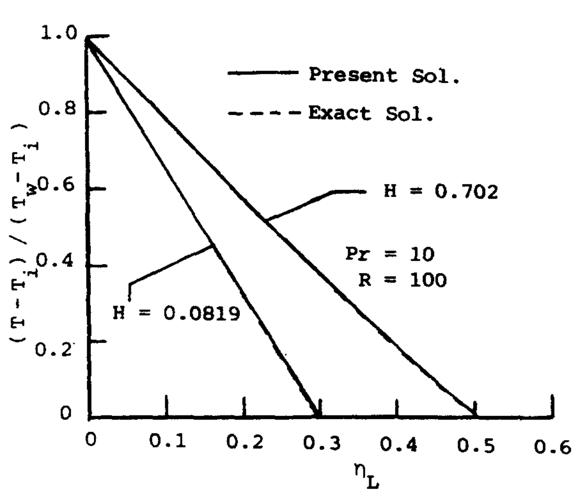

Figure 5. Temperature profiles.

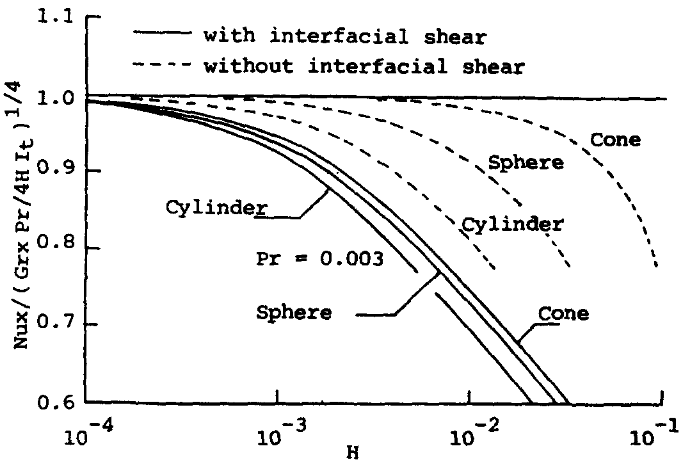

Figure 6. Similar film layers on cone, cylinder and sphere.

Calculations have been also carried out for the other similar flows, namely, the flow on a vertical cone as well as the stagnation flows on a horizontal circular cylinder and a sphere. The results are plotted altogether in figure 6 where the ordinate variable is chosen to have a common asymptote (unity) for the no-inertia limit. Therefore, for the direct comparison, the value read from the figure should be multiplied by the factor of $I_t^{-1/4}$ (namely, 1.236 for a cone, 1.075 for a cylinder and 1.278 for a sphere). The dashed lines in the figure correspond to the results obtained without considering the interfacial shear (Nakayama, Koyama & Ohsawa 1982). The interfacial shear has a significant effect on the heat transfer rate. The reduction in the heat transfer level seems considerable especially for the cone.

As for the application of the present integral procedure to the nonsimilar flows, the aforementioned step-wise iterative calculations have been performed for the condensation on a horizontal circular cylinder. The heat transfer rates averaged over the circumference of cylinder are shown in figure 7 along with the results obtained by Chen (1961) using a perturbation method. $\bar{h}$ in the ordinate variable is the overall heat transfer coefficient, and d is the diameter of cylinder.

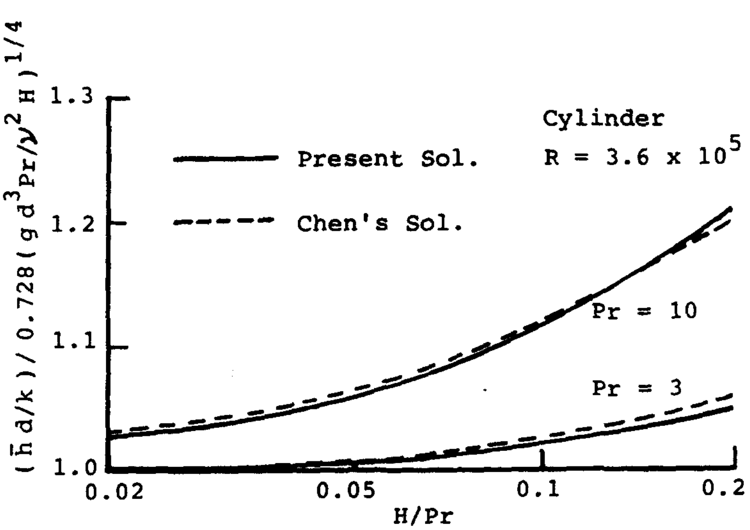

Figure 7. Heat transfer from a horizontal circular cylinder.

CONCLUDING REMARKS

Since the present analysis, unlike previous approximate analyses, takes full account of the inertia and convection effects, it is particularly suited for a speedy, and yet, accurate heat transfer estimate on the film condensation of low Prandtl number fluids such as liquid metals. An effort has been made to retain the density–viscosity ratio R as an additional parameter so that the film condensation at a high pressure may well be treated without loss of accuracy.

It is also interesting to note that, in the case of the saturated film boiling, the governing equations are identical to those presented here for the film condensation, except that the roles of liquid and vapor phases are interchanged. Thus, the present solution procedure is readily applicable for the saturated film boiling problems simply by substituting a small value into R which, then, should be defined as $(\rho\mu)_G/\rho\mu$.

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